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Analysis of Julia and Mandelbrot Sets Via Iterative Mappings 2020 Mathematical Subject Classification

M. I. Islamova[1,2*], R. R. Kutlumuratov[2], A. J. Ismailov[3]

[123] Tashkent State University of Economics, Tashkent, Uzbekistan
[1] Urgench State University

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*Correspondence: M. I. Islamova
Email: m.islamova@gmail.com

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Abstract: This paper provides a comprehensive analysis of Julia and Mandelbrot sets through the lens of iterative mappings in the complex plane. We investigate the dynamic behavior of quadratic polynomials, focusing on the fundamental connection between the connectedness of Julia sets and parameters within the Mandelbrot set. Employing the escape-time algorithm, we present detailed computational visualizations and explore key characteristics like critical points, self-similarity, and the topological features of both sets. This paper analyzes filled Julia and Mandelbrot sets based on iterative mappings, including the behavior of fixed points and 2-periodic points. Each definition is rigorously explained from a mathematical perspective, and visual representations are presented via bifurcation curves and cobweb diagrams. 2010 Mathematical Subject Classification: Primary: 54H20, Secondary: MSC 2010: 37B25, 37C25, 37C27

Keywords: Fixed Point, Attractor, Repeller, Julia Sets, Mandelbrot Sets, Escape-Time Algorithm, Computational Visualization.

Introduction

The study of Julia and Mandelbrot sets has fascinated mathematicians and scientists for decades due to their intricate geometric structures and their connection to complex dynamics. These sets are generated through the iterative application of simple polynomial functions in the complex plane. The Mandelbrot set, in particular, serves as a parameter space that dictates the topological properties of the corresponding Julia sets. This paper aims to provide a detailed analysis of these sets using the framework of iterative mappings, exploring their mathematical properties and computational visualization.

Fractal sets are among the most intriguing and complex objects in mathematics. Julia and Mandelbrot sets are frequently generated through iterative mappings on the complex plane. An indepth study of their properties is essential for mathematical modeling, visualization, and theoretical analysis. In this article, we explore Julia and Mandelbrot sets based on certain iterative functions, analyze the associated trajectories, fixed and periodic points, and construct their bifurcation lines.

Methodology

The monograph [2], considers a one-dimensional case, i.e. a differential equation of the form $x^{\cdot} = F(\lambda,x)$, where $F : \mathrm{R}^k \times \mathrm{R} \to \mathrm{R}$, $(\lambda, x) \to 7\ F(\lambda, x)$, $F(\lambda,x) \in C^2, C^3$. The equilibrium with quadratic and cubic degeneracy is studied for it. In the case of cubic degeneracy, the complexity level occurs near the equilibrium point at zero. In this paper, it is shown that equilibrium cannot be achieved from the linear and second-order terms of the Taylor series expansion for F.

For arbitrary two-dimensional mappings $F_{c1c2}(x,y) : R^2 \to R^2$ let us recall the following definitions [5], [6], which we will rely on in our work, where $(x,y) \in R^2$ and $(c_1,c_2) \in R^2$.

Definition 1. The filled Julia set of a mapping $F_{c1c2}(x,y) : R^2 \to R^2$ is defined as the set of all points (x,y), which have bounded orbits with respect to this mapping.

$$K(F_{c_1c_2}) = \{(x,y) : F^n_{c_1c_2}(x,y) < \infty, n \to \infty\}.$$

Definition 2. Julia set is the common boundary of the filled Julia set

$$J(F_{c1c2}) = \partial K(F_{c1c2}).$$

Definition 3. Mandelbrot set M_{Fc1c2} of the of the mapping F is the set of all points (c_1,c_2) in the parameter plane the orbits of all critical points of this mapping are bounded.

Definition 4. An attractive fixed point of the mapping F_{c1c2} is the fixed point (x_0,y_0) of the mapping F_{c1c2} such that any value of (x,y) in the domain is close enough to (x_0,y_0) the iterated function sequence.

$$(x,y), F_{c1c2}(x,y), F_{c1c2}(F_{c1c2}(x,y)), F_{c1c2}(F_{c1c2}(F_{c1c2}(x,y))), \ldots$$

converges to (x_0,y_0). Attractive fixed points also called as attractors.

Definition 5. The basin of attractive fixed point (x_0,y_0) is the set of all (x,y) points which converge to the fixed point (x_0,y_0) under F_{c1c2}

$$B(x_0,y_0) = \{(x,y) : F^n_{c_1c_2}(x,y) \to (x_0,y_0), n \to +\infty\}$$

.

Definition 6. If the sequence $x_0, x_1, \ldots$ is a stationer sequence, then is called the fixed point of

$$fix(P) = \{x : P(x) = x\}.$$

Definition 7. If for the sequence $x_0, x_1, \ldots$ equality $P^m(x_0) = x_0$ is satisfied and it is not true for any natural number less than then x_0 is called periodic point of $P(x)$ with prime period m.

$$Per^m(P) = \{x : P^m(x_0) = x_0\}$$

.

Definition 8. Let $P(x)$ be a map on X and let p is fixed point $P(p) = p$. If all points sufficiently close to p are attracted to p, then p is called a attracting fixed point. More precisely, if there is an $\varepsilon > 0$ such that all x in the epsilon neighborhood $N_\varepsilon(p)$, $lim_{k\to\infty} f^k(x) = p$, then p is attractor.

Definition 9. Let be a map on X and let p is fixed point $P(p) = p$. If all points sufficiently close to p are repelled from p, then p is called a repelling fixed point. More precisely, if there is an $\varepsilon > 0$ such that all x in the epsilon neighborhood $N_\varepsilon(p)$ except for p itself eventually maps outside of $N_\varepsilon(p)$, then p is called repeller.

{

Result and Discussion

Consider the mapping

$(y' = x2 - ay \; x' = y^2 - bx$

. First, we find its fixed points by solving the following system:

$(x^2 - ay - y = 0 \; y^2 - bx - x = 0$

From the first equation, we solve for $y = \frac{x^2}{a+1}$, and substitute into the second to find the fixed points:

$$\left(\frac{x^2}{a+1}\right)^2 - bx - x = 0$$

$$\frac{x^4}{(a+1)^2} - bx - x = 0$$

$$x^4 - b(a+1)^2 x - (a+1)^2 x = 0$$

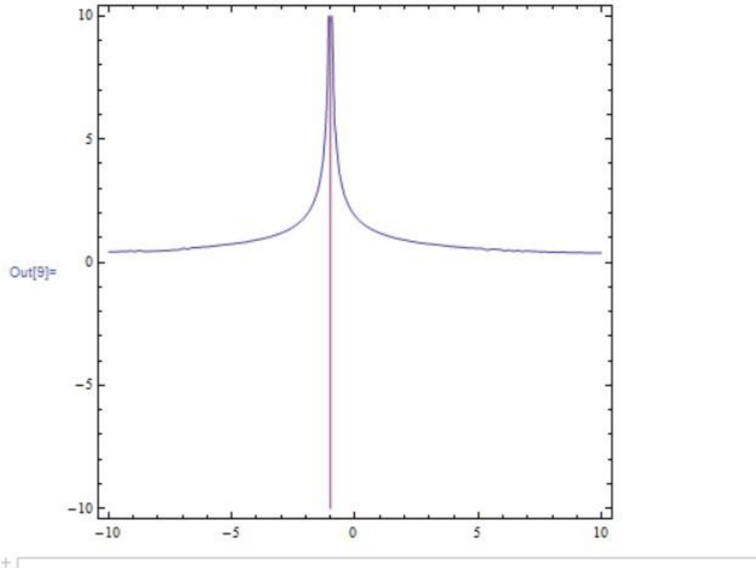

Figure 1. Figure a

$x\left(x^3 - b(a+1)^2 x - (a+1)^2\right) = 0$

$$x_1 = 0$$

$y_1 = 0$–*fixed points.*

$$x^3 - b(a+1)^2 x - (a+1)^2 = 0$$

Next, we need to find the solutions of this mapping. For that, we compute the resultant of the system, since the resultant is equal to the discriminant up to a multiplicative constant. If $f(x)$ and $f(x)$ are polynomials, their resultant is defined as:

$$D = \pm R(f(x), f'(x)) = \begin{pmatrix} 1 & 0 & 0 & -(a+1)^2 & 0 & 0 & 0 \\ 0 & 1 & 1 & -b(a+1)^2 & -(a+1)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -b(a+1)^2 & -(a+1)^2 & 0 \\ 0 & 0 & -b(a+1)^2 & 1 & 0 & -b(a+1)^2 & -(a+1)^2 \\ 3 & -b(a+1)^2 & -(a+1)^2 & 0 & 0 & 0 & 0 \\ 0 & 3 & -b(a+1)^2 & -(a+1)^2 & 0 & 0 & 0 \\ 0 & 0 & 3 & -b(a+1)^2 & -(a+1)^2 & 0 & 0 \end{pmatrix}$$

Hence,

$D = -(a+1)^4\left(4a^2b^3 + 8ab^3 + 4b^3 - 27\right)$

.

Discussion

Now we draw the bifurcation curves associated with the system. (Figure a)

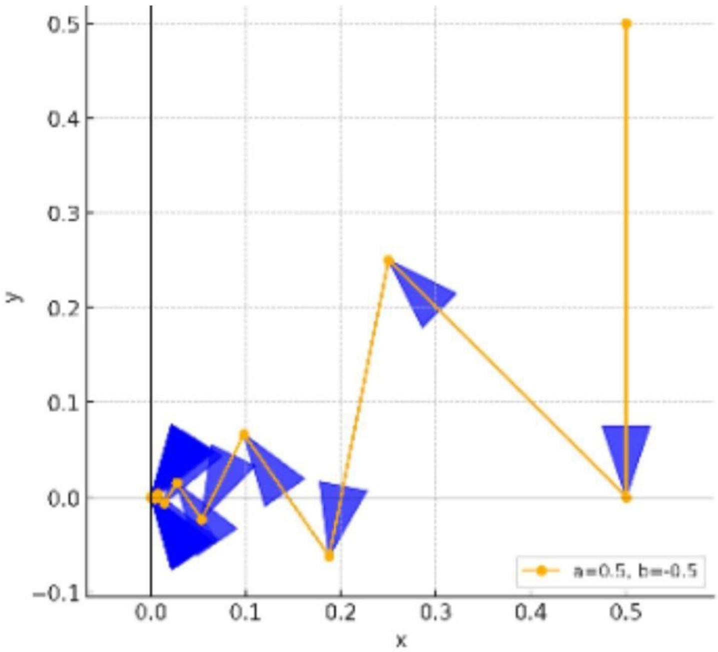

Figure 2. Figure 1

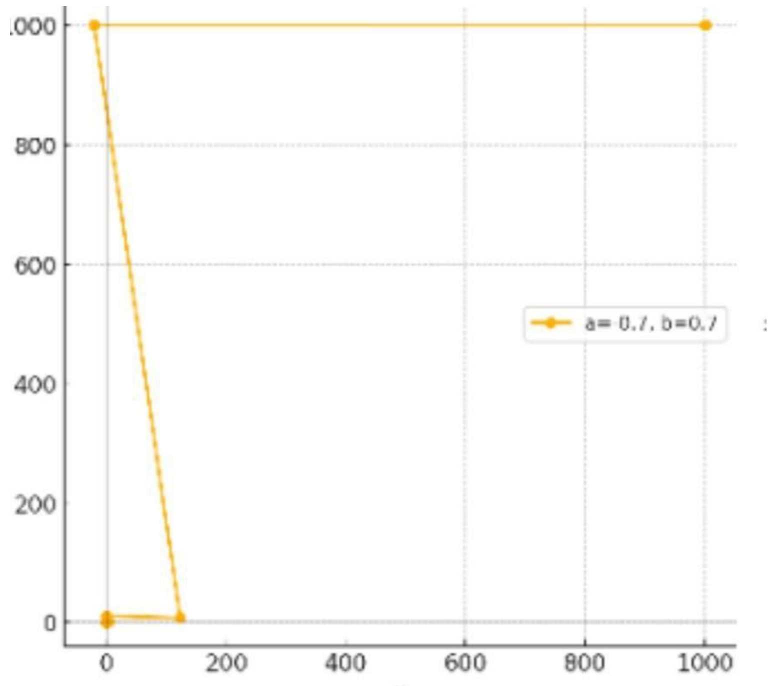

Figure 3. Figure 2

4. Numerical Simulations.

In this part of the paper, we investigate the filled Julia sets of our mapping and examine some interesting orbits within them.

1) For $a = 0.5$and $b = -0.5, x_0 = 0.5$, $y_0 = 0.5$(Figure 1)
2) For $a = -0.7$ and $b = 0.7$, $x_0 = -0.5$, $y_0 = 0.5$(Figure 2)
3) For $a = 0.9$and $b = -0.9$, $x_0 = 0.1$ and $y_0 = -0.1$ (Figure 3)

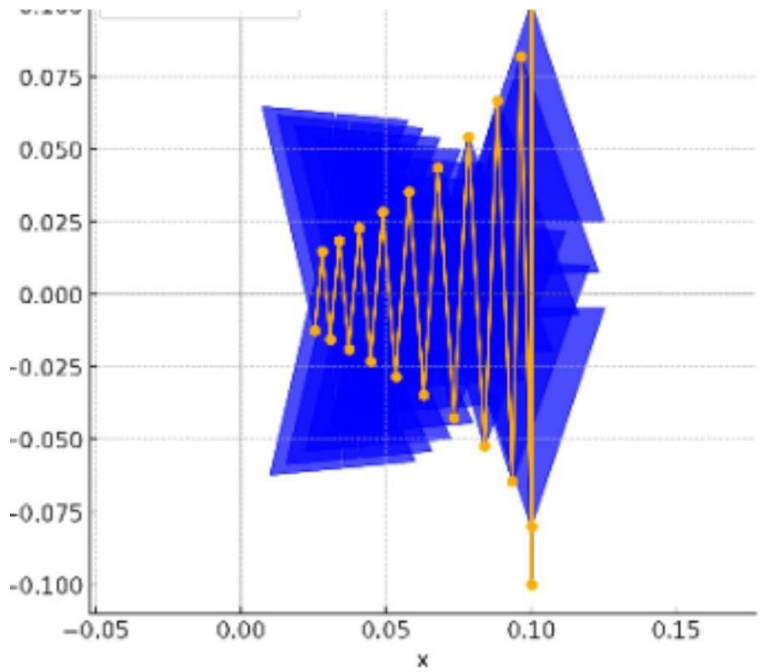

Figure 4. Figure 3

Analysis of the Mandelbrot Set

- Connectivity and Topology:
- Discuss the connectedness of the Mandelbrot set and its boundary.
- Explain the relationship between the Mandelbrot set and the bifurcations of Julia sets.
- Bulbs and Cardioids:
- Describe the main cardioid and the bulbs attached to it, explaining their mathematical significance.
- Relate the periods of the bulbs to the behavior of the corresponding Julia sets.
- Computational Visualization:
- Present a detailed image of the Mandelbrot set, generated using a computer program.
- Discuss the visual characteristics of the Mandelbrot set, such as its shape, the Misiurewicz points, and the hyperbolic components.
- Zooming into the Mandelbrot Set: Show a series of zoomed-in images of different regions of the Mandelbrot set, revealing the self-similar nature of the fractal.

Conclusion

As a result of the study, the filled versions of Julia and Mandelbrot sets, their trajectories, and bifurcation characteristics were identified. The parameter-dependence of the mappings and their dynamics were illustrated using cobweb diagrams. Analysis of 2-periodic points revealed the system's stability conditions and how they evolve with parameter changes. We constructed bifurcation curves that divide the parameter plane into four distinct regions, including two that there is not fixed points. These curves were shown to form the boundaries of the Mandelbrot set for the system under consideration. Additionally, through computational analysis, we generated a complete bifurcation diagram. For this system, we proved that the filled Julia set takes the shape of a quadrangle. Using numerical methods, we demonstrated that the filled Julia set contains an attractor and periodic points for some values of the parameter. In future work, we plan to explore its application as an biological model.

We have explored the relationship between the connectedness of Julia sets and the location of the parameter in the Mandelbrot set. The visualization techniques and computational methods employed have allowed us to characterize the complex dynamics and geometric properties of these fractal structures. The Julia and Mandelbrot sets serve as examples of how simple mathematical rules can lead to complex and beautiful patterns, making them an intriguing area of study in complex dynamics.

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